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A modified ϵ expansion for a hamiltonian with cubic point-group symmetry

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Abstract. The critical behaviour, in zero field above the critical temperature, of a hamiltonian with hypercubic point-group symmetry is studied in the framework of the ϵ expansion. An exponent associated with the anisotropy parameter Δ is calculated to order ϵ^3. It does not determine reliably whether or not Δ is an irrelevant variable in the critical region in three dimensions. The general structure of correlation functions is examined and found to be more complicated than in the isotropic case. It is suggested that these additional complications may be at least partially simplified by modifying the eigenvalue condition on the isotropic coupling constant to include the anisotropic coupling Δ.

1. Introduction

Universality has played a central role in recent theoretical developments in the description of critical phenomena (for a review see Wilson and Kogut 1973). Loosely speaking, it may be stated that the gross behaviour of, for example, all ferromagnets (to choose a definite nomenclature) is the same near their critical temperatures. Associated with this concept is the realization that one may expect a meaningful description in the critical region from a model which contains the essential features of the interactions in the system without necessarily containing all the detailed microscopic interactions.

One of the essential features of the interactions is the dimension n of the spin-density variable $s_i(x)$, $i = 1, 2, \ldots, n$, in terms of which the basic hamiltonian is $O(n)$ invariant and may be written

$$\frac{\mathscr{H}}{kT} = \int d^dx\left(\tfrac{1}{2}(\nabla s)^2 + \tfrac{1}{2}r_0 s^2 + \frac{u_0}{4!}(s^2)^2\right). \tag{1}$$

Here $s^2 = \Sigma_{i=1}^n s_i s_i$, d is the dimension of space and the relevant temperature dependence is contained in r_0 which is linear in temperature:

$$r_0 = a + bT. \tag{2}$$

The $O(n)$ invariance of this hamiltonian implies that one is considering an isotropic ferromagnet. This is certainly an idealization, and as an exercise in the study of universality, one may introduce a further term

$$\frac{\mathscr{H}_\Delta}{kT} = -\frac{\Delta}{4}\int d^dx s^4 \tag{3}$$

with $s^4 = \Sigma_{i=1}^n s_i^4$. This term reduces the symmetry to that of a hypercubic point group and must be present if the system has only that symmetry. We do not consider its microscopic origin but work in the spirit that it contains at least some of the essential features of a true microscopic hamiltonian.

In the framework of the renormalization group Wilson and Fisher (1972) looked at the fixed points of such a hamiltonian to order $\epsilon(\epsilon = 4-d)$ for $n = 2$ and Wegner (1972b) obtained critical exponents in the disordered phase above the critical temperature to order ϵ.

More recently, in their renormalization group study, Cowley and Bruce (1973) make clear that at least one other term may be necessary in the hamiltonian for the description of cubic systems (see also Fisher and Aharony 1973).

The equation of state and free energy has been studied for $\Delta > 0$ (Wallace 1973) in the limit of large spin-dimensionality n and the system shown to undergo a first order phase transition, with the spontaneous magnetization vanishing discontinuously by an amount dependent on the magnitude of anisotropy Δ. Far enough away from the transition temperature, the behaviour is essentially that of the isotropic model.

Whether or not such behaviour holds for a physical system with $n = 3$ cannot be determined with any confidence by the large n expansion method, and to this end we study in this paper the hamiltonian (3) in the framework of the ϵ expansion with n arbitrary. We consider only the behaviour above the transition temperature in zero external field and leave physical considerations and a study of the equation of state in the ϵ expansion, for a future paper.

The additional problems created by the term in equation (3) are well illustrated by the connected four-point correlation function at zero momentum and we restrict our attention to it:

$$u_{ijkl} = \frac{\int \mathrm{d}^d x\, \mathrm{d}^d y\, \mathrm{d}^d z \langle s_i(x) s_j(y) s_k(z) s_l(0)\rangle_c}{(\int \mathrm{d}^d x \langle s_i(x) s_i(0)\rangle)^4}. \tag{4}$$

Expectation values are calculated by Feynman graph expansion (see Wilson and Kogut 1973). The denominator removes external legs and self-energy insertions on them. For the same reason that there are two couplings at order s^4 in the hamiltonian, there are two independent terms, u_1 and u_2, in the tensor decomposition of u_{ijkl}:

$$u_{ijkl} = u_1(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + u_2\delta_{ijkl} \tag{5}$$

where

$$\delta_{ijkl} = \begin{cases} 1 & \text{if } i = j = k = l \\ 0 & \text{otherwise} \end{cases}$$

The outline of the paper is as follows. We consider Δ as small, $\Delta \ll u_0$, and in effect perform a systematic expansion in ϵ and Δ. This is not the attitude which one would normally adopt; normally Δ would be regarded as a coupling constant, fixed in order to produce scaling correlation functions. However, we are interested in the effect of small anisotropy on an essentially isotropic system and wish to have Δ free. In § 2 we consider the term in u_2 linear in Δ and evaluate it in the critical region to order ϵ^3. One obtains a critical exponent which determines whether or not Δ is an irrelevant variable. Unfortunately this exponent is not reliably determined in the ϵ expansion method for $n = 2$ or 3 and $\epsilon = 1$. In order to illustrate the structure of u_1 and u_2 for higher powers of Δ, we consider the large n limit of u_{ijkl} in § 3. There one sees the appearance of additional non-leading critical exponents differing from the most singular terms by order ϵ. It is suggested that these corrections to the scaling behaviour may be simplified by modifying the ϵ expansion by having a combination of u_0 and Δ constrained by an eigenvalue condition. In § 4 it is shown that this may be implemented to order Δ and to all orders in ϵ in the term u_1. Section 5 contains an incomplete discussion of order Δ^2 terms. Sufficient regularities are seen in these last two sections to merit further study.

2. A Δ-related exponent

In this section we consider the term in u_2 in equation (5) linear in Δ. This is the leading contribution to u_2 in an expansion in ϵ and Δ. We use the standard technique of introducing into the free hamiltonian a renormalized mass r which is the inverse of zero-field susceptibility. The mass counterterms are taken account of by subtracting self-energy insertions at zero momentum.

The graphs which contribute to order ϵ^3 in the ϵ expansion are shown in figure 1. It is to be understood that one sums over all possible positions of the Δ vertex in each graph in order to obtain the correct weight. One further point is that such graphs may also contribute to the term in u_1 linear in Δ and one must take care to separate the contributions to u_1 and u_2. This is easily done. Pictorially, it is convenient to draw modified Feynman graphs which also represent the way in which the spin indices i, j, k

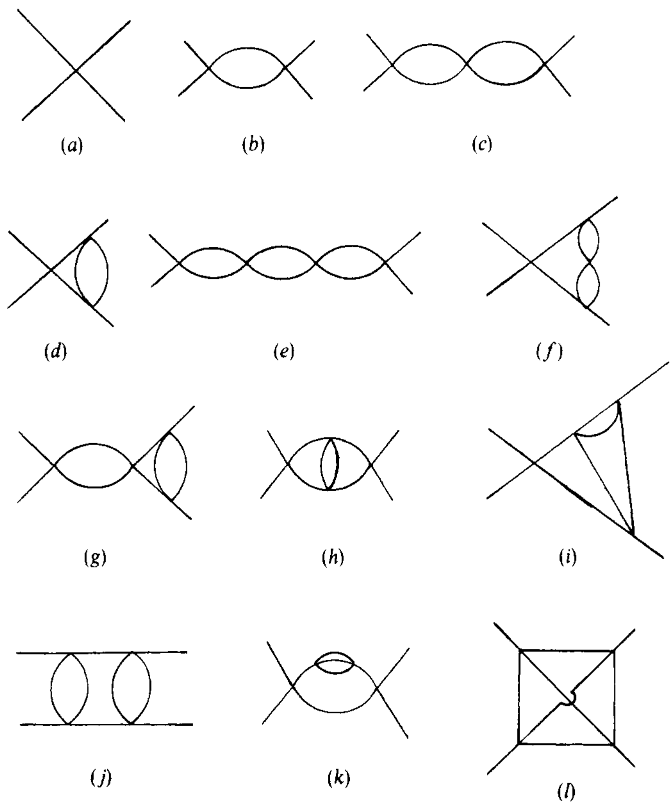

Figure 1. Graphs contributing to the four point correlation function to fourth order in perturbation theory.

and l flow through the diagram (see Wilson and Kogut 1973). The $\delta_{ij}\delta_{kl}$ graphs will always be 'separable', by pulling apart two pairs of external legs; the δ_{ijkl} graphs are 'non-separable', since the spin indices on the external legs must flow through the diagram to meet at the Δ vertex. The weights of the graphs which contribute to u_2 at order Δ are shown in table 1.

Table 1. Weights of graphs with one Δ vertex. The labelling is as in figure 1

(*a*)	6	(*e*)	$\frac{8}{3}$	(*i*)	$\frac{8}{3}(3n+22)$
(*b*)	12	(*f*)	$\frac{1}{3}(n^2+6n+40)$	(*j*)	$\frac{8}{3}(n+6)$
(*c*)	6	(*g*)	$\frac{4}{3}(n+10)$	(*k*)	$\frac{4}{3}(n+2)$
(*d*)	$2(n+14)$	(*h*)	$\frac{2}{3}(n+10)$	(*l*)	$\frac{8}{9}(n+14)$

We require the analytic expressions for such graphs in the critical region of small r so that the resultant expressions are correct to order ϵ^3. These calculations have been performed by Nickel (1973) in a calculation involving the isotropic hamiltonian (1) and his results are shown for completeness in table 2. In his calculations a cut-off is introduced by inserting a k^4 term in the denominator of propagators. Similar calculations have been performed using the Callan–Symanzik equation and renormalized perturbation theory (see Brézin *et al* 1973b) and the two methods give the same final results.

Finally, we note that the condition on u_0 which produces scaling in the ϵ expansion with a k^4 cut-off is given to order ϵ^3 by (Nickel 1973)

$$u = \frac{u_0 S_d \pi\epsilon}{2^d \pi^d \sin(\pi\epsilon/2)}$$

$$= \frac{12\epsilon}{n+8}\left[1+\left(\frac{9n+42}{(n+8)^2}-\frac{1}{2}\right)\epsilon+\left(\frac{1}{4}+\frac{(\pi^2-10\Lambda)(5n+22)+4(n+2)\Lambda/3}{2(n+8)^2}\right.\right.$$
$$\left.\left.+\frac{3n^3+160n^2+1192n+2632-12(5n+22)(n+8)\zeta(3)}{(n+8)^4}\right)\epsilon^3\right] \tag{6}$$

where π^2 and Λ are constants which never appear in final results, $\zeta(3) \simeq 1{\cdot}20$, and S_d is the surface area of a unit sphere in d dimensions. With our convention on normalization of the anaytic expressions in table 2, a factor of u should be inserted for each u_0 vertex in a diagram.

With this information, it is straightforward to reconstruct the scaling behaviour of the u_2 vertex. We find

$$u_2 \propto \Delta r^{\alpha_2} \tag{7}$$

where

$$\alpha_2 = \frac{6\epsilon}{n+8}-\frac{3(n^2+4n+28)\epsilon^2}{(n+8)^3}$$
$$-\frac{3\epsilon^3}{2(n+8)^5}[n^4+36n^3+172n^2+336n-1184-16(n+8)(n^2+7n+46)\zeta(3)]. \tag{8}$$

Table 2. Required analytic expressions for graphs with one Δ vertex. The labelling is as in figure 1

(*a*)	1
(*b*)	$\frac{1}{4}[\ln r+2-\frac{1}{4}\epsilon(\ln^2 r+2\ln r)+\frac{1}{24}\epsilon^2(\ln^3 r+3\ln^2 r)]$
(*c*)	$\frac{1}{16}[\ln^2 r+4\ln r+4-\frac{1}{2}\epsilon(\ln^3 r+4\ln^2 r+4\ln r)]$
(*d*)	$\frac{1}{64}[2\ln^2 r+4\ln r-8-2\pi^2+16\Lambda-\epsilon(\ln^3 r+2\ln^2 r+(2-4\Lambda)\ln r)]$
(*e*)	$\frac{1}{64}(\ln^3 r+6\ln^2 r+12\ln r)$
(*f*)	$\frac{1}{64}(\frac{1}{3}\ln^3 r+\ln^2 r-2\Lambda\ln r)$
(*g*)	$\frac{1}{128}[\ln^3 r+4\ln^2 r+(-\pi^2+8\Lambda)\ln r]$
(*h*)	$\frac{1}{64}[\frac{1}{3}\ln^3 r+\ln^2 r+(-4-\pi^2+10\Lambda)\ln r]$
(*i*)	$\frac{1}{128}[\frac{1}{3}\ln^3 r+(-4-\pi^2+8\Lambda)\ln r]$
(*j*)	$\frac{1}{64}(\frac{1}{3}\ln^3 r+\ln^2 r-2\Lambda\ln r)$
(*k*)	$\frac{1}{256}[\ln^2 r+(4-\frac{8}{3}\Lambda)\ln r]$
(*l*)	$\frac{3}{32}\zeta(3)\ln r$

One checks also that all required exponentiation takes place. This may not be too surprising since one is really looking only at the correlation function

$$u_2\delta_{ijkl}\sim\int\langle s^4(\omega)s_i(x)s_j(y)s_k(z)s_l(0)\rangle$$

evaluated with the isotropic hamiltonian (1); the s^4 term is playing only the 'geometrical' role of producing the appropriate tensor form.

It is particularly interesting to compare the above results with the leading term in the isotropic tensor function. According to scaling laws (Wilson and Kogut 1973) this is given by

$$u_1\propto\epsilon r^{\alpha_1} \tag{9}$$

$$\alpha_1=\frac{\epsilon-2\eta}{2-\eta}=\frac{\epsilon}{2}-\frac{(n+2)\epsilon^2}{2(n+8)^2}+\frac{(n+2)\epsilon^3}{4(n+8)^2}-\frac{3(3n^2+20n+28)\epsilon^3}{(n+8)^4}. \tag{10}$$

If α_2 is greater than α_1, Δ is an irrelevant variable because it will always appear associated with a larger power of r than the isotropic leading term. Then the effect of the anisotropy will be completely negligible except in quantities where the contribution of the isotropic interactions vanishes identically, such as the u_2 term, and the transverse susceptibility in zero field below the critical temperature (Goldstone's theorem).

If α_2 is less than α_1, Δ is not irrelevant and will become the dominating interaction if r becomes small enough even though $\Delta\ll u_0$. This is a very clear example of the way that small parameters may produce strong effects in the critical region. Of course the form of the free energy may be such that r never becomes vanishingly small. For example for $\Delta>0$ when n is large enough (Wallace 1973) the system never reaches $r=0$, but undergoes a first order phase transition to the ordered magnetic state; at this point the $O(\Delta)$ effects are the same order of magnitude as the leading isotropic terms.

It is disappointing that the ϵ expansion does not determine the sign of $\alpha_2-\alpha_1$ with any reliability. One finds

$$\alpha_2-\alpha_1 = \frac{(4-n)\epsilon}{2(n+8)}-\frac{(5n^2+14n+152)\epsilon^2}{2(n+8)^3}-\frac{(n+2)\epsilon^3}{4(n+8)^2}$$
$$-\frac{3\epsilon^3}{2(n+8)^5}[n^4+30n^3+84n^2-40n-1632-16(n+8)(n^2+7n+46)\zeta(3)]. \qquad (11)$$

For $n = 3$ this gives

$$\alpha_2-\alpha_1 = 0{\cdot}045\epsilon-0{\cdot}089\epsilon^2+0{\cdot}139\epsilon^3.$$

The radius of convergence of this series does not appear to extend to $\epsilon = 1$. It is a matter of conjecture whether the radius of convergence is nonzero or whether the series is asymptotic.

3. Structure of higher corrections in Δ

Such straightforward exponentiation of logarithms to produce scaling behaviour does not take place in other terms involving Δ. For example in the isotropic vertex u_1, the term of order Δ arises first in figure 1(*b*) which gives an analytic form $\epsilon\Delta \ln r$. It is not *a priori* clear how this $\ln r$ should be exponentiated to produce scaling behaviour.

In this section we use the limit $n \to \infty$ to elucidate the structure of the higher order terms in Δ in the ϵ expansion. The features it illustrates may not be completely general but we have not discovered any discrepancies yet. A more convincing, but perhaps not so clear, method would involve use of the renormalization group.

The leading contribution to u_1 for n large and $\Delta = 0$ has been discussed by Wilson (1973); we include it for completeness. It is given as a geometric sum of a stream of bubbles and the result is

$$u_1(r) = \tfrac{1}{3}u_0(1+\tfrac{1}{6}nu_0 I(0,r))^{-1} \qquad (12)$$

where

$$I(q,r) = \int_p (p^2+r)^{-1}[(\boldsymbol{p}+\boldsymbol{q})^2+r]^{-1} \qquad (13)$$

$$\int_p = (2\pi)^{-d}\int_0^\Lambda \mathrm{d}^d p$$

and Λ is some cut-off. Evaluating $I(0,r)$ for $r \ll \Lambda^2$, one finds

$$I(0,r) = \tfrac{1}{2}(2\pi)^{-d}S_d[\Gamma(2-\tfrac{1}{2}\epsilon)\Gamma(\tfrac{1}{2}\epsilon)r^{-\epsilon/2}-2\Lambda^{-\epsilon}/\epsilon]+\mathrm{O}(r^{1-\epsilon/2}/\Lambda^2). \qquad (14)$$

In general the diverging $r^{-\epsilon/2}$ term in this expression controls the behaviour in the critical region ($r \to 0$) and one obtains

$$u_1(r) = \frac{u_0}{1-nu_0 a\Lambda^{-\epsilon}+nu_0 b r^{-\epsilon/2}} \to \frac{r^{\epsilon/2}}{nb}, \qquad r \to 0 \qquad (15)$$

where a and b are constants given in equation (14). However, there is in general an infinite series of powers of r of the form $r^{m\epsilon/2}(m = 1, 2, \ldots)$ in this expression. One can eliminate all these nonleading terms if one fixes u_0 by the condition

$$nu_0 a\Lambda^{-\epsilon} = 1 \tag{16}$$

and produce only the leading term (15). The condition (16) is the equivalent in the large n limit of expression (6).

The leading contribution and $1/n$ corrections to the $O(\Delta)$ term in u_2 are given respectively in figures 2(*a*) and 2(*b*) (which contains the typical stream of bubbles). Their analytic expressions may be evaluated by standard methods (Wilson 1973) and one obtains

$$u_2 \propto \Delta r^{\alpha_2}$$

with

$$\alpha_2 = +\frac{12\sin(\frac{1}{2}\pi\epsilon)}{\pi n(1-\frac{1}{2}\epsilon)}\frac{\Gamma(2-\epsilon)}{\Gamma^2(1-\frac{1}{2}\epsilon)}. \tag{17}$$

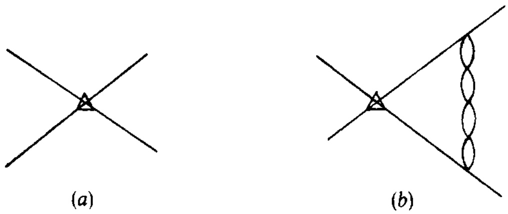

Figure 2. The leading and $1/n$ corrections to the correlation function u_2 with one Δ vertex.

Further it is readily seen that there are corrections to this leading behaviour involving an infinite series in $r^{\epsilon/2}$ but that all these correction terms vanish if u_0 obeys the condition (16). This illustrates the calculation of the previous section and indeed one easily checks that expressions (8) and (17) agree to $O(\epsilon^3, n^{-1})$.

As an aside, we note also that expressions (8) and (17) are in agreement with the scaling law

$$\psi = (\delta-1)(1+\alpha_2-\alpha_1)$$

with ψ as given by Wallace (1973).

Let us look now at the higher order terms in Δ in u_2. At order Δ^2 the leading term is figure 1(*b*), which gives $54\Delta^2 I(0, r)$. This expression gives the expected leading behaviour proportional to $r^{-\epsilon/2}$ and a single non-leading term proportional to r^0, from the existence of the cut-off Λ (see equation (14)). Unlike expression (15), there is only one additional non-leading term and it *cannot* be eliminated by any eigenvalue condition on u_0. Thus the order Δ^2 term in u_1 must be expected to exponentiate to at least two powers of r, differing by order ϵ.

More generally, one may consider the term of order Δ^l in u_2. The reader may convince himself that this is given by the sum of all graphs with $l\,\Delta$ vertices; all insertions of u_0 vertices are down by order $1/n$. The analytic expressions for these graphs will give the expected leading behaviour proportional to $r^{-(l-1)\epsilon/2}$ with all less singular powers of $r^{-\epsilon/2}$ up to r^0. Thus the terms at order Δ^l must exponentiate to at least l powers of r, differing by order ϵ.

Although the situation now looks rather complicated, it is suggestive that at a given order l in Δ all the non-leading powers of r are those associated with lower powers of Δ. This conjecture would imply that u_2 has the structure

$$u_2 \sim \Delta f_1(\Delta) r^{\alpha_2} + \sum_{l=2}^{\infty} \Delta^l f_l(\Delta) r^{\alpha_1 + l(\alpha_2 - \alpha_1)}$$

with $f_l(0) =$ constant. This situation, where the full scaling behaviour is given in terms of functions of Δ rather than Δ alone, is analogous to the discussion in Wegner (1972a). A very limited verification of this structure in the ϵ expansion is given in § 5.

Let us return now to the isotropic term, u_1. The order Δ correction to this vertex is given by the standard stream of bubbles with one Δ insertion. To order Δ, one finds

$$u_1 = \tfrac{1}{3}u_0(1+\tfrac{1}{6}u_0 nI(0,r))^{-1} + \tfrac{1}{6}u_0\Delta I(0,r)(12+nu_0 I(0,r))(1+\tfrac{1}{6}u_0 nI(0,r))^{-2}. \tag{18}$$

With the condition (16) on u_0, one obtains two powers of r at order Δ, r^0 and r^ϵ. However in this expression one has the freedom of modifying the condition (16) on u_0 by a term of order Δ in order to eliminate the non-leading r^ϵ behaviour. Using expressions (14), (15) and (16) it is straightforward to show that the condition

$$u_0 = \frac{\Lambda^\epsilon}{na} + \frac{18\Delta}{n} \tag{19}$$

eliminates the non-leading r^ϵ term from expression (18) and leaves the correct scaling power of r, exactly as in the term linear in Δ in u_2.

The generalization of equation (19) in the ϵ expansion is considered in the following section. It is shown there that such a generalized eigenvalue condition does exist at order Δ; whether this result can be generalized to higher orders in Δ remains to be seen.

Let us summarize this section. It is clear from the anisotropic term u_2 that one cannot expect the ln r terms in the ϵ expansion to exponentiate to single powers of r; at order Δ^l, l powers of r will be present each differing by order ϵ. It is suggested that the non-leading powers of r are precisely those associated with lower powers of Δ. The structure in the isotropic function u_1 is more complicated, but may reduce to the same as u_2 by the use of a generalized eigenvalue condition involving both u_0 and Δ.

4. The generalized eigenvalue condition at order Δ

The relevant contribution to the isotropic term u_1 of tree and one-loop diagrams is given to $O(\epsilon^2, \Delta)(\Delta = O(\epsilon))$ by

$$u_1 = \tfrac{1}{3}u_0 - (48\pi^2)^{-1}[\tfrac{1}{6}u_0^2(n+8) - 6\Delta u_0] \ln r. \tag{20}$$

We look for a solution for c in the expression

$$u_0 = \frac{48\pi^2\epsilon}{n+8} + c\Delta \tag{21}$$

which produces the correct exponentiation at order Δ, that is a term $\Delta r^{6\epsilon/(n+8)}$ in order to agree with expression (8). One requires

$$\frac{c}{3} + (48\pi^2)^{-1}\left(\frac{c}{3}48\pi^2\epsilon - 6\frac{48\pi^2\epsilon}{n+8}\right) \ln r \propto 1 + \frac{6}{n+8}\epsilon \ln r$$

which gives

$$c = \frac{18}{n+2}. \tag{22}$$

This result has at least two necessary features: (i) it agrees for large n with the previous expression (19) and (ii) for $n = 1$, $(\phi^2)^2$ and ϕ^4 interactions are identical so that Δ is no longer an independent control variable. Indeed when $n = 1$, equations (21) and (22) give

$$u_0 - 6\Delta = \frac{16\pi^2\epsilon}{3}.$$

This is precisely the linear combination of u_0 and Δ appearing in the hamiltonians (1) and (3) and ensures that the equations (21) and (22) are correct for $n = 1$.

Moreover it is encouraging that, given the existence of an eigenvalue condition

$$u_0 = f(\epsilon) \tag{23}$$

which ensures scaling behaviour for the term in u_2 linear in Δ, then the condition

$$u_0 = f(\epsilon) + \frac{18\Delta}{n+2} \tag{24}$$

ensures the *same* scaling behaviour for the term in u_1 linear in Δ, *to all orders in* ϵ.

The proof is as follows. Consider the general expressions for u_1 and u_2

$$u_1 = \sum_l (u_0^l A_l - l\Delta u_0^{l-1} B_l + \ldots) \tag{25a}$$

$$u_2 = \sum_l (-l\Delta u_0^{l-1} C_l + \ldots). \tag{25b}$$

The terms in brackets represent the sum over all graphs at a given order l in ϵ. We exhibit explicitly that there is also a polynomial in Δ in each expression. A_l, B_l and C_l represent the sum over all appropriate graphs at a given order l. The factors of l are for convenience in counting.

Now, for any graphs g with a given topology of l vertices, one has

$$18A_l^g = C_l^g + (n+2)B_l^g. \tag{26}$$

Proof: Graphs C_l^g and B_l^g have one Δ vertex inserted where there is a u_0 vertex in A_l^g. In C_l^g, the spin indices of the Δ vertex must be traceable out to the external legs, since u_2 arises in the tensor δ_{ijkl}. In A_l^g one has a similar contribution in which the spin on the lines of the u_0 vertex which has replaced the Δ vertex of C_l^g can be traced out to the external legs. Such graphs in A_l^g are down by a factor of eighteen—a factor of six (because u_0 appears with $(4!)^{-1}$ and Δ with $\frac{1}{4}$ in the hamiltonian) and a factor of three difference in the contractions to form the $\delta_{ij}\delta_{kl}$ and δ_{ijkl} tensors. However in A_l^g a pair of spin indices of this u_0 vertex may also eventually contract: contractions of this type are contained in B_l^g. The relevant factor $18/(n+2)$ arises because the contraction of a pair of lines from a $(\phi^2)^2$ vertex produces a factor of $(n+2)$ whereas from a ϕ^4 vertex one obtains only a factor of three.

To paraphrase this argument: take a given u_0 vertex in graph A_l^g. There are two classes of contractions which the legs of this vertex can make: (*a*) their spin indices can be traced to the external legs and (*b*) at least one pair of the spin indices eventually contracts. Graphs C_l^g reproduce class (*a*); graphs B_l^g reproduce class (*b*). The argument is the same in spirit to that used by Balian and Toulouse (1973).

Equation (26) ensures that when the condition (24) is substituted into equations (25) then for each graph the coefficients of Δ in (25*a*) and (25*b*) are equal up to a unique factor and hence if exponentiation takes place in the u_2 function it also takes place in u_1, with the same exponent in both cases.

We have not dotted all the i's in this argument but in conjunction with the fact that we have checked it explicitly to order ϵ^3 it should be convincing enough.

5. Some results at order Δ^2

In this section we consider briefly the structure of the $O(\Delta^2)$ terms in the ϵ expansion. The results are in agreement with the picture obtained by the large n expansion in § 3 but calculations have not yet been made to sufficient level to be regarded as completely convincing.

First we consider the anisotropic term u_2, up to graphs with three vertices (figures 1(*a*) to 1(*d*)). There are two distinct types of contributions; those from graphs with two Δ vertices and those from graphs with one Δ vertex in conjunction with the eigenvalue condition (24). Since we are interested principally in studying the exponentiation of the $\ln r$ terms we retain only $\Delta^2 \ln r$ and $\Delta^2\epsilon \ln^2 r$ terms. The result is

$$u_2 = \frac{27\Delta^2(n-2)}{8\pi^2(n+2)}\left(\ln r - \frac{\epsilon}{4}\ln^2 r + \frac{9\epsilon}{n+8}\ln^2 r\right). \tag{27}$$

For definiteness, we assume that these logarithms exponentiate to two powers of r, the leading power of which has the correct scaling behaviour, ie in lowest order in ϵ,

$$u_2 = f\Delta^2(r^{(16-n)\epsilon/(2n+16)} - r^{\alpha_3\epsilon}). \tag{28}$$

It is trivial to compare the ϵ expansion of expression (28) with (27) and obtain

$$\alpha_3 = \frac{6}{n+8} \tag{29}$$

and

$$f = \frac{27(n-2)(n+8)}{4\pi^2(n+2)(n-4)\epsilon}. \tag{30}$$

Equation (29) supports our previous statement in § 3 that the second power of r in equation (28) is precisely the power of r associated with the term in u_2 linear in Δ. Note that these terms are $O(\Delta^2/\epsilon)$, that is $O(\epsilon)$.

We have also checked this structure for u_2 for graphs with four vertices. If the logarithms are to exponentiate to the two powers of r, generalizing equation (28), then one finds two constraints, one from the leading logarithms and one from the next to leading logarithm terms. Both of these constraints are satisfied when α_3 is the exponent associated with the $O(\Delta)$ term in u_2, as given in equation (8).

As an aside we remark that we have shown that all contributions of $O(\Delta^2)$ to u_2 will contain a factor of $(n-2)$ to all orders in ϵ. The proof is in the same spirit as the proof of the condition (24) in § 4. The fact that these $O(\Delta^2)$ contributions to u_2 vanish for $n = 2$ need not imply they are small for $n = 3$, since a factor of $n-4$ appears in the denominator of expression (30).

Finally, we have looked at the $O(\Delta^2)$ term in u_1 to order ϵ^2, in the same spirit as above. One may find a correction of order (Δ^2/ϵ) to the condition (24) to ensure that the

$\ln r$ and $\ln^2 r$ terms exponentiate to the same two powers of r as in equations (28) and (29). We quote the result

$$u_0 = f(\epsilon) + \frac{18\Delta}{n+2} - \frac{72(n-1)(n+8)}{16\pi^2 n(n+2)^2}\frac{\Delta^2}{\epsilon}(1+O(\epsilon)). \tag{31}$$

Note that this last correction of $O(\Delta^2/\epsilon)$ vanishes for $n = 1$, as it should according to the arguments following equation (22).

However, it should be clear that such higher-order extensions of the eigenvalue condition require further study.

6. Conclusion

The original aim of this work was to estimate the exponent α_2 related to the anisotropy parameter, Δ. Up to order ϵ^3, it is found that one cannot reliably determine whether or not Δ is an irrelevant variable in the critical region for a system with a two- or three-dimensional 'spin' vector in three space dimensions.

It became clear that the general structure of the correlation functions is also of considerable interest; generally speaking, as one looks at higher powers of Δ, there appear more and more corrections to the leading behaviour associated with the given power of Δ. These corrections involve powers of inverse susceptibility r, differing by order ϵ from the leading power and make the reconstruction of the critical behaviour increasingly more complicated. However, we have tried to suggest that the situation is not as black as might at first appear because there are sufficient regularities in these correction terms to allow further progress.

We conclude by remarking on the significance of the relevance or otherwise of Δ in the critical region. It is interesting to consider three possibilities.

(i) Δ is irrelevant ($\alpha_2 > \alpha_1$). Then the isotropic fixed point (corresponding to $\Delta = 0$, u_0 as in equation (6)) is attractive and the exponent $\alpha_2 - \alpha_1$ governs only corrections to scaling behaviour which are negligible close enough to the critical point.

(ii) Δ is relevant ($\alpha_2 < \alpha_1$) and the system undergoes a second order phase transition. Then the exponent $\alpha_2 - \alpha_1$ controls the crossover from isotropic to a new behaviour. The inverse susceptibility r becomes arbitrarily small at the critical point and in order to obtain the true critical behaviour of for example the correlation function u_1 one must know the asymptotic behaviour of the power series of u_1 in $\Delta r^{\alpha_2-\alpha_1}$.

(iii) Δ is relevant and the system undergoes a first order phase transition. At the transition, r has some finite value and the behaviour just above the transition might possibly be determined fairly accurately by some low order terms in the expansion in powers of Δ.

The expansion about the isotropic fixed point enables one in principle to determine whether or not Δ is relevant but does not discriminate readily between situations (ii) and (iii). One may turn then to a study of the renormalization group (Wilson and Kogut 1973) or the Callan–Symanzik equation (see Brézin *et al* 1973a and references therein). At order ϵ one finds that the isotropic fixed point is attractive for $n < 4$, as expected. For $n > 4$, the isotropic fixed point is repulsive, and there exists an attractive fixed point with $\Delta = O(\epsilon)$ and $\Delta < 0$. Therefore if one has a system with a small negative Δ (and $n > 4$) it will have critical behaviour determined by this second fixed point, corresponding to possibility (ii) above. For $\Delta > 0$ (and $n > 4$) the behaviour is governed by neither of these fixed points; it is in this situation that an expansion of the form considered in this paper might be useful. For $n = 4$, by working to order ϵ^2 one finds that the isotropic fixed point is repulsive, again as expected since the ϵ^2 term in $\alpha_2 - \alpha_1$ (situation (ii)) is always negative.

Thus the use of the renormalization group confirms and supplements the information in this paper. These and other results will be reported in greater detail by Bervillier (1973).

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Note added in proof. Recursion formulae (see Wilson and Kogut 1973) have also been used by Aharony (to be published) to study the nature of the phase transition in possibility (ii) of our conclusion.

References

Balian R and Toulouse G 1973 *Phys. Rev. Lett.* **30** 544–6
Bervillier C 1973 in preparation
Brézin E, le Guillou J C and Zinn-Justin J 1973a *Phys. Rev.* D **8** 434–40
Brézin E, le Guillou J C, Zinn-Justin J and Nickel B G 1973b *Phys. Lett.* **44A** 227–8
Cowley R A and Bruce A D 1973 *J. Phys.* C: *Solid St. Phys.* **6** L191–2
Fisher M E and Aharony A 1973 *Phys. Rev. Lett.* **30** 559–62
Nickel B G 1973 to be published
Wallace D J 1973 *J. Phys.* C: *Solid St. Phys.* **6** 1390–404
Wegner F J 1972a *Phys. Rev.* B **5** 4529–36
—— 1972b *Phys. Rev.* B **6** 1891–3
Wilson K G 1973 *Phys. Rev.* D **7** 2911–26
Wilson K G and Fisher M E 1972 *Phys. Rev. Lett.* **28** 240–3
Wilson K G and Kogut J 1973 *Phys. Rep.* to be published